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Split string formalism and the closed string vacuum

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Abstract: The split string formalism offers a simple template upon which we can build many generalizations of Schnabl's analytic solution of open string field theory. In this paper we consider two such generalizations: one which replaces the wedge state by an arbitrary function of wedge states, and another which generalizes the solution to conformal frames other than the sliver.

Keywords: String Field Theory, Tachyon Condensation.

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Contents

1. Introduction

In a remarkable paper [1] Schnabl constructed, for the first time, a well-behaved analytic solution to the equations of motion of open bosonic string field theory (OSFT).[1] The solution represents the most basic and important field configuration for the open string: the vacuum where the D-brane has decayed and there are no open strings left. By now it has been confirmed numerically and analytically that the potential energy of the vacuum matches the D-brane tension [1, 3, 4] and that the vacuum supports no open strings [5], in accordance with Sen's conjectures [6].

Given the complexity of the solution in its various forms, it is instructive to strip it down to its bare bones in the hope generalizing its structure and codifying essential lessons in the search for other solutions. As noticed by Okawa [3] perhaps the simplest expression of Schnabl's solution comes in the split string formalism [7, 8]. There, the solution can be expressed algebraically in terms of "matrix products" of three string fields K, B and c. These three fields are postulated to satisfy six simple identities eq. (2.12). The expression,

$$\Psi = F(K)c\frac{KB}{1-F(K)^2}cF(K) \tag{1.1}$$

is then guaranteed to satisfy the equations of motion for any string field F which depends only on K. For Schnabl's solution, $F(K)$ is a wedge state, specifically the "square root" of the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum.

[1] For a nice review, see ref. [2].

In this paper we investigate this basic structure in the search for generalizations of Schnabl's solution. We consider two types of generalization: the first generalizes the choice of the field $F(K)$, and the second searches for new realizations K, B, c such that the simple identities which make eq. (1.1) work are still satisfied. This second generalization, we will see, is essentially equivalent to the choice of a projector and its associated conformal frame. Schnabl's solution is based on the sliver projector, the state obtained by repeated multiplication of the $\mathrm{SL}(2,\mathbb{R})$ vacuum with itself. Aside from some general comments, our discussion of generic $F(K)$ will be limited; we will analyze the simplest example where F is allowed to be an arbitrary wedge state. The completely general case will be considered in a companion paper [9] (henceforth, (II)).

This paper is organized as follows. In section 2 we develop the basic framework and explain how Schnabl's solution maps between the split string formalism and the conformal field theory representation. In section 3 we slightly generalize $F(K)$, allowing it to be an arbitrary power of the $\mathrm{SL}(2,\mathbb{R})$ vacuum; we show that these solutions are related by a simple midpoint-preserving reparameterization symmetry. In section 4, we generalize Schnabl's solution to other projector conformal frames. The major challenge here is to find a workable conformal field theory representation of these solutions. Our strategy is to define a new conformal frame, the "strip frame," where the equivalent of wedge states are described as infinite strips in the complex plane. Manipulation of the solution and calculation of the energy then proceeds in exact analogy with Schnabl's solution. We illustrate this for the butterfly projector and some of its multi-winged cousins. We end with some conclusions.

While our work was nearing completion, the paper ref. [10] appeared which has significant overlap with some of our results. We hope however that this paper offers a useful perspective on these types of generalization.

2. Split string formalism

The split string formalism (SSF) is a formal approach to computations in OSFT which represents the star product as a "matrix product" of half string functionals [7, 8]. The basic idea is quite intuitive, though the concrete implementation can become quite technical. For our purposes, however, all we need is the basic philosophy: that the action of an operator on the string field Ψ can represented as star multiplication of Ψ with an appropriate state. In this way, all computations in OSFT reduce to computations of the star product ("matrix product") and the BPZ inner product ("trace").

The starting point for the SSF is the overlap condition, which states that the vertex identifies the left and right halves of the string for neighboring functionals, defining a kind of "matrix product." In particular, if $\langle\Psi_1, \Psi_2 * \ldots\rangle_N$ is the N-string vertex, the string position coordinate $x(\sigma)$ satisfies,

$$\begin{aligned}\langle\Psi_1, x(\sigma)\Psi_2 * \ldots\rangle_N &= \langle x(\pi-\sigma)\Psi_1, \Psi_2 * \ldots\rangle_N \quad \sigma\in[0,\tfrac{\pi}{2}]\\ x(\sigma) &= X(z,\bar{z})|_{z=e^{i\sigma}}\end{aligned}$$

and so on cyclically. This can be generalized to an arbitrary local operator $A(z)$. Let

$$\phi = e^{i\sigma} \quad \sigma \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

so that $A(\phi)$ acts on the left half of the string at σ. Then the overlap condition states,[2]

$$\langle \Psi_1, A(\phi)\Psi_2 * \ldots \rangle_N = \langle A^*(\phi)\Psi_1, \Psi_2 * \ldots \rangle_N \tag{2.1}$$

where A^* is the BPZ dual of A. If A is a dimension h primary, then

$$A^*(\phi) = \phi^{-2h} A\left(-\frac{1}{\phi}\right) \tag{2.2}$$

Another useful object is the identity string field $|I\rangle$, which is postulated to satisfy

$$\langle I, \Psi_1 * \Psi_2 * \ldots \rangle_{N+1} = \langle \Psi_1, \Psi_2 * \ldots \rangle_N \tag{2.3}$$

The existence of $|I\rangle$ immediately gives a prescription for implementing the action of operators on the star algebra in terms of the star algebra itself. For example we can write,

$$A(\phi)\Psi = \mathcal{A} * \Psi \tag{2.4}$$

where the field $\mathcal{A}$ is defined,

$$\mathcal{A} = A(\phi)|I\rangle \tag{2.5}$$

The proof is,

$$\begin{aligned} \langle \chi, A(\phi)\Psi \rangle_2 &= \langle A^*(\phi)\chi, \Psi \rangle_2 = \langle A^*(\phi)\chi, I * \Psi \rangle_3 = \langle \chi, (A(\phi)I) * \Psi \rangle_3 \\ &= \langle \chi, \mathcal{A} * \Psi \rangle_3 \end{aligned} \tag{2.6}$$

A similar argument shows that for operators acting on the right half of the string,

$$A^*(\phi)\Psi = (-1)^{A\Psi}\Psi * \mathcal{A} \tag{2.7}$$

Also,

$$A^*(\phi)|I\rangle = A(\phi)|I\rangle \tag{2.8}$$

Note that the above manipulations become a little delicate when A acts on the midpoint $\phi = \pm i$. Unless A has dimension zero, evaluating $A(i)$ in the vertex creates vanishing or divergent factors which can yield some surprises.

Let us make some remarks about conventions. In our definition of the vertex eq. (2.1) the right string half of the previous functional is identified with the left string half of the subsequent functional. Sometimes the opposite convention is used, for example in ref. [3]; these conventions are related by a twist. Also, the left half of the string $\sigma \in [0, \frac{\pi}{2}]$ is mapped to the *right* half semi-circle $e^{i\sigma}$ in the complex plane. To avoid confusion, we will always refer to the image of the left half of the string in the complex plane as the *positive* side, and the image of the right as the *negative* side.

[2] If A is Grassmann odd we must multiply by the appropriate sign from anticommutation.

JHEP05(2007)083

The only other thing we need from the SSF is notation. When writing the star product we drop the star, and when calculating the BPZ inner product we replace the bracket with a trace. Thus,

$$\Psi * \Phi = \Psi\Phi \qquad \langle \Psi, \Phi \rangle = \mathrm{Tr}(\Psi\Phi) \tag{2.9}$$

The identity string field is denoted by 1. We may then manipulate string fields like ordinary matrices, taking care of Grassmann parity. For example,

$$\mathrm{Tr}(\Psi\Phi) = (-1)^{\Psi\Phi}\mathrm{Tr}(\Phi\Psi) \tag{2.10}$$

The physical string field is Grassmann odd.

2.1 Solution in the SSF

As noticed by Okawa [3], we can construct a "split string" solution to the field equations given any three string fields,

$$\begin{aligned} K &= \text{Grassmann even, gh\# } = 0 \\ B &= \text{Grassmann odd, gh\# } = -1 \\ c &= \text{Grassmann odd, gh\# } = 1 \end{aligned} \tag{2.11}$$

which satisfy the following algebraic properties:

$$\begin{aligned} Bc + cB = 1 \qquad & KB - BK = 0 \qquad & B^2 = c^2 = 0 \\ dK = 0 \qquad & dB = K \qquad & dc = cKc \end{aligned} \tag{2.12}$$

where $d = Q_B$ is the BRST operator.[3] These relations imply that K, B, c freely generate a subalgebra of the star algebra which is closed under the action of the BRST operator. It is then simple to show [3] that the state,

$$\Psi = F(K)c\frac{KB}{1-F^2(K)}cF(K) \tag{2.13}$$

formally satisfies,

$$d\Psi + \Psi^2 = 0 \tag{2.14}$$

for any field $F(K)$ which can be written purely in terms of star products of K alone. For the appropriate choice of K, B, c and F, this is in fact Schnabl's solution.[4]

Let us see how the identities eq. (2.12) can be explicitly realized. We will suppose K, B, c take the form

$$\begin{aligned} K &= K_L^v|I\rangle \\ B &= B_L^v|I\rangle \\ c &= c^v(1)|I\rangle \end{aligned} \tag{2.15}$$

[3] In the split string formalism we might like to express d as an inner derivation $d\Psi = D\Psi - (-1)^{\epsilon(\Psi)}\Psi D$ for some string field D. Formally this is possible [11], though there can be subtleties because d acts on the midpoint. We will not find it necessary or useful to express d as an inner derivation.

[4] It seems possible that eq. 2.13 describes all acceptable solutions within the subalgebra generated by K, B, c. We have found another solution $\Psi = Fc\frac{K}{F}$ but it seems singular and does not satisfy the reality condition. If other solutions exist it could be of great interest.

JHEP05(2007)083

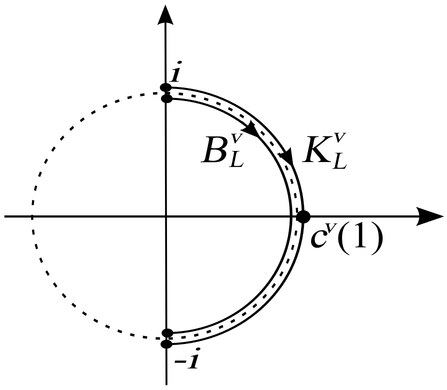

Figure 1: Operators defining the fields K, B, c.

where,

$$K_L^v = \int_L \frac{d\xi}{2\pi i} v(\xi) T(\xi) \qquad B_L^v = \int_L \frac{d\xi}{2\pi i} v(\xi) b(\xi) \qquad c^v(1) = -\frac{1}{v(1)} c(1) \tag{2.16}$$

and $v(\xi)$ is an arbitrary holomorphic vector field, subject to certain conditions which we will explain in a moment. The contour L is taken over the "left half" of the string, that is, along the positive semi-circle connecting $\xi = +i$ to $\xi = -i$ (see 1). We will use R to denote the contour from i to $-i$ on the negative semi-circle. From these definitions and eq. (2.6) it is straightforward to verify eq. (2.12). For example,

$$Bc + cB = (B_L^v c^v(1) + c^v(1) B_L^v)|I\rangle = 1 \tag{2.17}$$

and

$$dB = (Q_B B_L^v + B_L^v Q_B)|I\rangle = K_L^v |I\rangle = K \tag{2.18}$$

where we used the fact that the identity is BRST closed.

The vector field $v(\xi)$ is subject to two conditions: reality and regularity. The reality condition follows from the requirement that K, B, c be real string fields, which in split string language means they are "self-adjoint" matrices, $K^\dagger = K$ etc. If (and only if) this condition holds, the solution Ψ is real. For a state of the form $\mathcal{O}|I\rangle$, the reality condition reads,

$$\langle \mathcal{O} I, \chi\rangle = \langle I|\mathcal{O}^\dagger|\chi\rangle$$

Using eq. (2.8) and the fact that $|I\rangle$ is real, we can write the left hand side,

$$\langle \mathcal{O}^* I, \chi\rangle = \langle I, \mathcal{O}\chi\rangle = \langle I|\mathcal{O}|\chi\rangle \tag{2.19}$$

Thus $\mathcal{O}$ must be a hermitian operator. Imposing this on eq. (2.16) requires,

$$\overline{v(\xi)} = \bar{\xi}^2 v\left(\frac{1}{\bar{\xi}}\right) \tag{2.20}$$

Note that in our definition of c we took the ghost insertion to be precisely on the real axis. Actually, if the insertion is taken off the real axis the identities eq. (2.12) are still satisfied, but then c and the resulting solution would not be real.

The second condition on $v(\xi)$ is a regularity condition: v must vanish at the midpoint,

$$v(\pm i) = 0 \tag{2.21}$$

As mentioned earlier, when representing operators as string fields in the SSF, it is best to avoid operators which act on the midpoint. Furthermore, as explained in ref. [12], left/right decompositions of energy momentum charges $\oint vT$ are anomalous unless $v(\pm i) = 0$. Probably solutions which fail to satisfy eq. (2.21) are undefined.

Following the philosophy of refs. [1, 12] it is useful to think of the operator K_L^v as arising from the energy momentum zero mode in a nonstandard conformal frame. Let $z = f(\xi)$ define the local coordinate for a surface state represented on the upper half plane. The energy momentum zero mode in this conformal frame is,

$$\mathcal{L}_0 = \oint \frac{dz}{2\pi i} zT(z) = \oint \frac{d\xi}{2\pi i}\frac{f(\xi)}{f'(\xi)}T(\xi) \tag{2.22}$$

Its BPZ conjugate is,

$$\mathcal{L}_0^* = \oint \frac{d\xi}{2\pi i}\left(-\xi^2\frac{f(-1/\xi)}{f'(-1/\xi)}\right)T(\xi) \tag{2.23}$$

Up to a proportionality we may identify,

$$K_L^v = (\mathcal{L}_0 + \mathcal{L}_0^*)_L \tag{2.24}$$

where the subscript L denotes integrating the corresponding current over the contour L pictured in figure (1[5]). Thus,

$$v(\xi) = \frac{f(\xi)}{f'(\xi)} - \xi^2\frac{f(-1/\xi)}{f'(-1/\xi)} \tag{2.25}$$

Notice that for real, twist invariant conformal frames,

$$\overline{f(\xi)} = f(\bar{\xi}) \quad f(-\xi) = -f(\xi)$$

the reality condition eq. (2.20) is automatically satisfied. Moreover, the regularity condition eq. (2.21) follows if

$$f(\pm i) = \infty \tag{2.26}$$

that is, if $f(\xi)$ is the conformal frame for a *projector*[13]. Thus, the importance of projectors in Schnabl's solution is ultimately related to the fact that it can be expressed using the split string formalism.

In summary, we have a generic set of solutions to the string field equation, characterized by an arbitrary field $F(K)$ and choice of projector with conformal frame $f(\xi)$. In the rest of this paper we study the implications of these solutions.

[5]Note that $K_L^v = -sL_L^+$ in the notation of ref. [12]. The sign comes from our definition of "left," which is chosen so that K_L^v creates, rather than destroys, infinitesimal strips in the sliver coordinate frame.

2.2 Schnabl's solution

Though it may be possible to realize the solution Ψ in terms of a matrix or Moyal product of open string functionals, our strategy will be to map the solutions to the conformal field theory representation where it is clearer how to define Ψ in terms of correlators.

In this capacity, let us review Schnabl's solution in the form eq. (2.13) and its relation to the conformal field theory representation. Schnabl's solution corresponds to the choice,

$$f(\xi) = \tan^{-1}\xi \tag{2.27}$$

$$F(K) = \exp(\tfrac{1}{2}K) = \Omega^{1/2} \tag{2.28}$$

The first equation says the solution is formulated in the sliver conformal frame [1, 13]. The second equation says that F is the square root of the $\mathrm{SL}(2,\mathbb{R})$ vacuum Ω. We also have,

$$\begin{aligned} v(\xi) &= \frac{\pi}{2}(1+\xi^2) \\ K^v_L &= \frac{\pi}{2}(L_1+L_{-1})_L \quad B^v_L = \frac{\pi}{2}(b_1+b_{-1})_L \quad c^v(1) = -\frac{1}{\pi}c(1) \end{aligned} \tag{2.29}$$

Let us recall why F in the form eq. (2.28) is the square root of the vacuum, because it's important. A wedge state Ω^α can be represented as a correlation function on the cylinder $C_{\frac{\pi}{2}(\alpha+1)}$. This cylinder can be represented as a strip $-\frac{\pi}{4} < \Re(z) < \frac{\pi\alpha}{2} + \frac{\pi}{4}$ with its ends identified. Specifically,

$$\langle \Omega^\alpha, \chi\rangle = \langle f\circ\chi(0)\rangle_{C_{\frac{\pi(\alpha+1)}{2}}} \tag{2.30}$$

with $\chi(0)$ the vertex operator of the state χ and f as in eq. (2.27). The derivative of Ω^α with respect to α is the proportional to the variation of the correlator with respect to the cylinder's circumference. Alternatively, this variation can be computed by inserting a "Hamiltonian" into the correlator which creates an infinitesimal strip of worldsheet parallel to the imaginary axis, as shown in figure 2. Thus,

$$\left\langle \frac{d}{d\alpha}\Omega^\alpha, \chi\right\rangle = \frac{\pi}{2}\frac{d}{d(\frac{\pi(\alpha+1)}{2})}\langle f\circ\chi(0)\rangle_{C_{\frac{\pi(\alpha+1)}{2}}} = \frac{\pi}{2}\left\langle f\circ\chi(0)\int_{i\infty}^{-i\infty}\frac{dz}{2\pi i}T(z)\right\rangle_{C_{\frac{\pi(\alpha+1)}{2}}} \tag{2.31}$$

We now absorb the $\int T$ into the local coordinate on the *negative* side of the puncture, and map back to the unit disk. The result is,

$$\left\langle \frac{d}{d\alpha}\Omega^\alpha, \chi\right\rangle = \left\langle f\circ\left(\chi(0)\int_R \frac{d\xi}{2\pi i}\frac{\pi}{2}(1+\xi^2)T(\xi)\right)\right\rangle_{C_{\frac{\pi(\alpha+1)}{2}}} = \langle \Omega^\alpha, K^v_R\chi\rangle \tag{2.32}$$

Noting $K^{v\,*}_R = K^v_L$, this gives,

$$\frac{d}{d\alpha}\Omega^\alpha = K^v_L\Omega^\alpha \tag{2.33}$$

Integrating and using eq. (2.6)

$$\Omega^\alpha = \exp(\alpha K^v_L)|I\rangle = \exp(\alpha K) \tag{2.34}$$

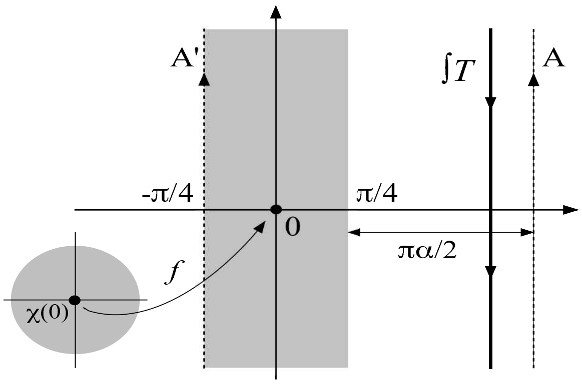

Figure 2: Correlator defining the derivative of the wedge state Ω^α. The dashed vertical lines A,A' are cut and sewn together to form a cylinder.

implying eq. (2.28).[6]

To calculate the D-brane energy using Schnabl's solution, it is necessary to regulate the form eq. (2.13) in a particular way [1]. Specifically, the solution is rewritten

$$\Psi = \lim_{N\to\infty}\left(\sum_{n=0}^{N}\psi'_n - \psi_N\right) \tag{2.35}$$

where,

$$\psi'_n = FcBKF^{2n}cF \qquad \psi_n = \frac{1}{\gamma}FcBF^{2n}cF \tag{2.36}$$

and γ is a constant associated with the definition of F, as we will explain in section 3.[7] This amounts to a truncated Taylor expansion of our previous formula, modulo the mysterious ψ_N piece which vanishes when contracted with Fock space states [1]. In the following we will *assume* that this regulator is correct, regardless of the choice of F or conformal frame.[8] Eq. (2.35) is also convenient for translating to CFT language, since when $F = \Omega^{1/2}$ the ψ'_ns can be computed as correlators on the cylinder with particular insertions [3]. To get the insertions and their normalizations correctly, it is useful to note that for an primary operator $A(\phi)$ acting on the left side of the string,

$$\langle A(\phi)\Omega^\alpha,\chi\rangle = \frac{1}{(1+\phi^2)^h}\left\langle A\left(\frac{\pi(\alpha+\frac{1}{2})}{2}+iy\right) f\circ\chi(0)\right\rangle_{C_{\frac{\pi(\alpha+1)}{2}}} \tag{2.37}$$

where,

$$y = \frac{1}{4}\tanh^{-1}\sin\sigma \tag{2.38}$$

[6] While the relation $\exp(\alpha K_L^v)|I\rangle = \exp(\alpha K)$ appears formal, we have checked that it can be given a precise meaning in the Moyal representation of the star product, at least in the matter sector. Ghosts would be trickier to handle because one has to treat zero modes carefully and cancel off divergent determinants. Perhaps the recent results of ref. [14] would help in this regard.

[7] For Schnabl's solution $\gamma = 1$, as shown in [1, 3, 4].

[8] Note that for general F, $\psi'_n \neq \frac{d}{dn}\psi_n$.

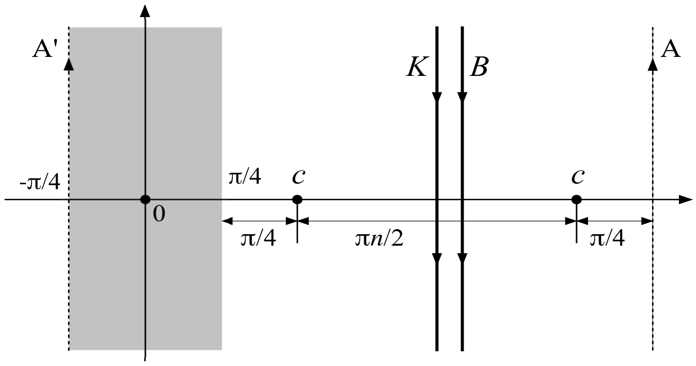

Figure 3: Conformal field theory representation of the state ψ'_n.

In particular, the operator is mapped to the positive boundary of the strip defining the wedge state, $\Re(z) = \frac{\pi(\alpha+\frac{1}{2})}{2}$. The string endpoint $\phi = 1$ ($\sigma = 0$) is mapped to the real axis, and the midpoint $\phi = i$ ($\sigma = \frac{\pi}{2}$) is mapped to infinity, as expected. Using this mapping, the definition of K, B and c, and the CFT gluing prescriptions for the star product, one can rewrite the ψ'_ns as correlators on the cylinder:

$$\langle \psi'_n, \chi \rangle = \left\langle c\left(\frac{\pi(n+1)}{2}\right) KBc\left(\frac{\pi}{2}\right) f \circ \chi(0) \right\rangle_{C_{\frac{\pi(n+2)}{2}}} \tag{2.39}$$

where we have defined the contour insertions,

$$K = \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} T(z) \qquad B = \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} b(z) \tag{2.40}$$

hopefully not to be confused the fields K, B introduced earlier (see figure 3). This is the well-known result of ref. [3][9]

2.3 Energy: general considerations

Ultimately we are interested in calculating the energy and giving a physical interpretation of these new solutions. Thus, it's worthwhile seeing what can be said about the energy before we specify F, f and try to evaluate correlators.

Assuming the equations of motion, the action evaluated on the (regulated) solution is,

$$\begin{aligned} E &= -\frac{1}{6}\langle \Psi, Q_B \Psi \rangle \\ &= -\frac{1}{6} \lim_{N\to\infty} \left[\sum_{m,n=0}^{N} \langle \psi'_m, Q_B \psi'_n \rangle - 2 \sum_{m=0}^{N} \langle \psi'_m, Q_B \psi_N \rangle + \langle \psi_N, Q_B \psi_N \rangle \right] \end{aligned} \tag{2.41}$$

[9] Actually, perhaps the simplest formulation is to define the fields K, B, c as cylinder correlators from the start. For example,

$$\langle c, \chi \rangle = -\left\langle c\left(\frac{\pi}{2}\right) f \circ \chi(0) \right\rangle_{C_{\frac{\pi}{2}}}$$

Translating between CFT and the split string formalism then becomes trivial. However, generalizing this to arbitrary projector frames requires new ingredients, to be explained in section 4.

A crucial step in evaluating this comes from the realization that, for Schnabl's ψ'_ns, the following "diagonal" sum vanishes,

$$\sum_{k=0}^{n} \langle \psi'_{n-k}, Q_B \psi'_k \rangle = 0 \tag{2.42}$$

Because of this, the action only receives contributions from $\langle \psi_m, Q_B \psi_n \rangle$ for large $n+m$, where the correlators simplify [1, 3, 4]. Furthermore, for the pure gauge solutions of [1], the large $n+m$ terms manifestly don't contribute, so eq. (2.42) implies their energies vanish.

In ref. [1], eq. (2.42) was demonstrated by explicit evaluation of $\langle \psi'_m, Q_B \psi'_n \rangle$. As we will argue, this sum actually vanishes in general, regardless of the choice of F and conformal frame. This could be demonstrated directly with the proper manipulations invoking the identities eq. (2.12) and OSFT axioms, but we found it difficult to construct a simple proof in this manner. Instead, we utilize an observation of Okawa [3] that Schnabl's solution can be recast in a form which is (naively) pure gauge. Define the state,

$$\Phi = FBcF \tag{2.43}$$

A little computation with eq. (2.12) shows that,

$$(d\Phi)\Phi^n = FcKBF^{2n}cF = \psi'_n \tag{2.44}$$

Let us define a one parameter family of solutions,

$$\Psi_\lambda = \lambda F c \frac{KB}{1-\lambda F^2} cF = \sum_{n=0}^{\infty} \lambda^{n+1} \psi'_n \tag{2.45}$$

The real parameter λ could be absorbed into the definition of F, but let us keep it there for convenience. Plugging in eq. (2.44) gives,

$$\Psi_\lambda = \lambda(d\Phi)\frac{1}{1-\lambda\Phi} = (1-\lambda\Phi) d\left(\frac{1}{1-\lambda\Phi}\right) \tag{2.46}$$

This is in the form of a pure gauge solution, with gauge parameter

$$e^\Lambda = \frac{1}{1-\lambda\Phi} \tag{2.47}$$

Note in particular that,

$$\frac{d}{d\lambda}\Psi_\lambda = d\left(\frac{d\Lambda}{d\lambda}\right) + \left[\Psi_\lambda, \frac{d\Lambda}{d\lambda}\right] \tag{2.48}$$

so the variation of Ψ_λ with respect to λ is a gauge variation. Since by construction the action is gauge invariant, easy to show that,

$$\frac{d}{d\lambda}\mathrm{Tr}[\Psi_\lambda d\Psi_\lambda] = 0 \tag{2.49}$$

JHEP05(2007)083

But recalling eq. (2.45) we can rewrite this as,

$$
\begin{aligned}
0 &= \frac{d}{d\lambda} \sum_{m,n=0}^{\infty} \lambda^{m+n+2} \langle \psi'_m, Q_B \psi'_n \rangle \\
&= \sum_{n=0}^{\infty} (n+2) \lambda^{n+1} \sum_{k=0}^{n} \langle \psi'_{n-k}, Q_B \psi'_k \rangle
\end{aligned} \tag{2.50}
$$

implying eq. (2.42). Note that we did not need to assume anything about the solution beyond the identities eq. (2.12) and the OSFT axioms.

3. Arbitrary wedge states

In this section we investigate a simple generalization of Schnabl's solution which allows the field F to be an arbitrary wedge state:

$$
\begin{aligned}
F(K) &= e^{\gamma K/2} \\
&= \Omega^{\gamma/2}
\end{aligned} \tag{3.1}
$$

where $\gamma \in [0, \infty]$ is a constant defining a one parameter family of solutions. Note that the second line only follows when K is in the sliver frame eq. (2.29), which we will assume for simplicity — though the analogous generalization exists for any projector frame. More general choices of F will be considered in (II).

The main effect of eq. (3.1) is that the strip with insertions defining ψ'_n in figure 3 is "scaled" by a factor of γ. To see how this effects the solution, consider, generally speaking, two wedge states with insertions related by a scaling, Φ and Φ_γ. We can represent Φ as a strip $-\frac{\pi\alpha}{2} + \frac{\pi}{4} < \Re(z) < \frac{\pi\alpha}{2} - \frac{\pi}{4}$ in the complex plane, with local operators,

$$
\phi_1(z_1), \quad \phi_2(z_2), \quad \ldots \tag{3.2}
$$

placed inside. The "scaled" state Φ_γ is a strip $\gamma(-\frac{\pi\alpha}{2} + \frac{\pi}{4}) < \Re(z) < \gamma(\frac{\pi\alpha}{2} - \frac{\pi}{4})$ with insertions,

$$
\phi'_1(z'_1), \quad \phi'_2(z'_2), \quad \ldots \tag{3.3}
$$

placed inside, where

$$
z'_i = \gamma z_i \quad \phi'_i(z'_i) = s_\gamma \circ \phi_i(z_i) \quad s_\gamma(z) = \gamma z \tag{3.4}
$$

As explained in ref. [1], Φ can be given an explicit operator representation,

$$
|\Phi\rangle = \left(\frac{1}{\alpha}\right)^{\mathcal{L}_0^*} \left(\frac{1}{\alpha}\right)^{\mathcal{L}_0} \tilde{\phi}_1(z_1) \tilde{\phi}_2(z_2) \ldots |\Omega\rangle \tag{3.5}
$$

where,

$$
\tilde{\phi}_i(z_i) = f^{-1} \circ \phi_i(z_i), \qquad f(z) = \tan^{-1} z \tag{3.6}
$$

We now consider the contraction between Φ and a Fock space state $|\chi\rangle = \tilde{\chi}(0)|\Omega\rangle$:

$$
\langle \Phi, \chi \rangle = \left\langle \phi_1(z_1) \phi_2(z_2) \ldots \chi\left(-\frac{\pi\alpha}{2}\right) \right\rangle_{C_{\pi\alpha}} \tag{3.7}
$$

Scaling the correlator by a factor of γ,

$$\langle \Phi, \chi \rangle = \left\langle \phi_1'(z_1')\phi_2'(z_2')\ldots \chi'\left(-\frac{\pi\gamma\alpha}{2}\right)\right\rangle_{C_{\pi\gamma\alpha}} \tag{3.8}$$

We can see Φ_γ embedded in this, but unfortunately the scaled local coordinate has the wrong size to be a Fock space state. Nevertheless, this can be regarded as the contraction of Φ_γ with another wedge state χ_γ whose strip extends from $-\frac{\pi\gamma}{4} < \Re(z) < \frac{\pi\gamma}{4}$ and with a single insertion $\chi'(0)$:

$$\begin{aligned} \langle \chi_\gamma, \zeta \rangle &= \langle \chi'(0)\zeta\,(-\tfrac{\pi(\gamma+1)}{4})\rangle_{C_{\frac{\pi(\gamma+1)}{2}}} \\ |\chi_\gamma\rangle &= \left(\frac{2}{\gamma+1}\right)^{\mathcal{L}_0^*}\left(\frac{2}{\gamma+1}\right)^{\mathcal{L}_0}\tilde{\chi}'(0)|\Omega\rangle \end{aligned} \tag{3.9}$$

Thus,

$$\langle \Phi, \chi \rangle = \langle \Phi_\gamma, \chi_\gamma \rangle \tag{3.10}$$

Let us further simplify eq. (3.9). Note,

$$\begin{aligned} \tilde{\chi}'(0) &= f^{-1}\circ s_\gamma \circ \chi(0) = (f^{-1}\circ s_\gamma \circ f)\circ \tilde{\chi}(0) \\ &= \gamma^{\mathcal{L}_0}\tilde{\chi}(0)\gamma^{-\mathcal{L}_0} \end{aligned} \tag{3.11}$$

The last equality follows from the fact that $\mathcal{L}_0$ is the dilatation generator in the sliver frame. Thus eq. (3.9) becomes,

$$|\chi_\gamma\rangle = \left(\frac{2}{\gamma+1}\right)^{\mathcal{L}_0^*}\left(\frac{2\gamma}{\gamma+1}\right)^{\mathcal{L}_0}|\chi\rangle \tag{3.12}$$

eq. (3.10) then implies,

$$\Phi_\gamma = \left(\frac{1+\gamma}{2}\right)^{\mathcal{L}_0}\left(\frac{1+\gamma}{2\gamma}\right)^{\mathcal{L}_0^*}\Phi \tag{3.13}$$

With a little manipulation of the group algebra $[\mathcal{L}_0, \mathcal{L}_0^*] = \mathcal{L}_0 + \mathcal{L}_0^*$ [1, 12] this can be brought to the form,

$$\Phi_\gamma = \exp\left[\frac{1}{2}\ln\gamma(\mathcal{L}_0 - \mathcal{L}_0^*)\right]\Phi \tag{3.14}$$

where the operator $\mathcal{L}_0 - \mathcal{L}_0^*$ is a midpoint-preserving reparameterization generator. It is now simple to see that,

$$\psi_n^{(F=\Omega^{\gamma/2})} = \exp\left[\frac{1}{2}\ln\gamma(\mathcal{L}_0 - \mathcal{L}_0^*)\right]\psi_n^{(Schnabl)} \tag{3.15}$$

In particular, the factor $1/\gamma$ in eq. (2.36) arises from the nontrivial scaling dimension of the insertions. More generally, we will see in (II) that γ is given by the first order coefficient in the formal Taylor series expansion,

$$F(K) = 1 + \frac{\gamma}{2}K + \cdots \tag{3.16}$$

JHEP05(2007)083

At any rate, eq. (3.15) implies that the solutions with $F = \Omega^{\gamma/2}$ are related by midpoint preserving reparameterizations. Since these are symmetries of OSFT, it is clear that these solutions are physically equivalent.

It may be interesting to consider the limits $\gamma \to 0$ and $\gamma \to \infty$. The first limit formally gives an identity based solution, while the second gives something resembling the sliver. This suggests a connection with vacuum string field theory [15]—indeed in ref. [16] it was argued that vacuum string field theory could be derived from OSFT after an infinite midpoint reparameterization of the vacuum solution. Eq. (3.14) gives an exact one parameter family of solutions which realize such a reparameterization precisely. More work is necessary to understand these limits however, since a naive substitution $F = 1$ or $F = \Omega^\infty$ results in divergent expressions.

Given the importance of gauge-fixing in the original construction of the Schnabl's solution [1], it is worth deriving the gauge conditions satisfied by these solutions. Denoting the solutions by Ψ_γ, Schnabl's solution satisfies,

$$\mathcal{B}_0 \Psi_1 = 0 \tag{3.17}$$

where $\mathcal{B}_0$ is the b ghost zero mode in the sliver frame. Using the formula [1],

$$\left(\frac{1}{\alpha}\right)^{-\mathcal{L}_0^*} \mathcal{B}_0 \left(\frac{1}{\alpha}\right)^{\mathcal{L}_0^*} = \frac{1}{\alpha}\mathcal{B}_0 + \left(\frac{1}{\alpha} - 1\right)\mathcal{B}_0^* \tag{3.18}$$

we may reparameterize eq. (3.17) to get,

$$\left[\frac{\gamma+1}{2}\mathcal{B}_0 + \frac{\gamma-1}{2}\mathcal{B}_0^*\right]\Psi_\gamma = 0 \tag{3.19}$$

Interestingly, in general the gauge condition involves both creation and annihilation modes. For the identity-based solution $\gamma = 0$ the gauge fixing operator is a star algebra derivative:

$$(\mathcal{B}_0 - \mathcal{B}_0^*)\Psi_0 = 0 \tag{3.20}$$

Perhaps this indicates a general problem with constructing well-defined solutions in $*$-derivative gauges, such as $b_1 + b_{-1} = 0$. The $\gamma \to \infty$ "vacuum string field theory" limit gives the condition,

$$(\mathcal{B}_0 + \mathcal{B}_0^*)\Psi_\infty = 0 \tag{3.21}$$

This gauge also greatly simplifies computation of off-shell amplitudes, as was shown in ref. [18]. Interestingly, $\mathcal{B}_0 + \mathcal{B}_0^*$ happens to annihilate the sliver state, giving further evidence for the connection between vacuum string field theory and the $\gamma \to \infty$ limit.

4. Generalizing conformal frames

In this section we investigate the second generalization of Schnabl's solution suggested by the split string formalism: generalizing the choice of vector field $v(\xi)$, or equivalently, of a projector conformal frame $f(\xi)$. For definiteness we will take the analogue of Schnabl's $F(K)$,

$$F(K) = \exp(\tfrac{1}{2}K) \equiv \Xi^{1/2} \tag{4.1}$$

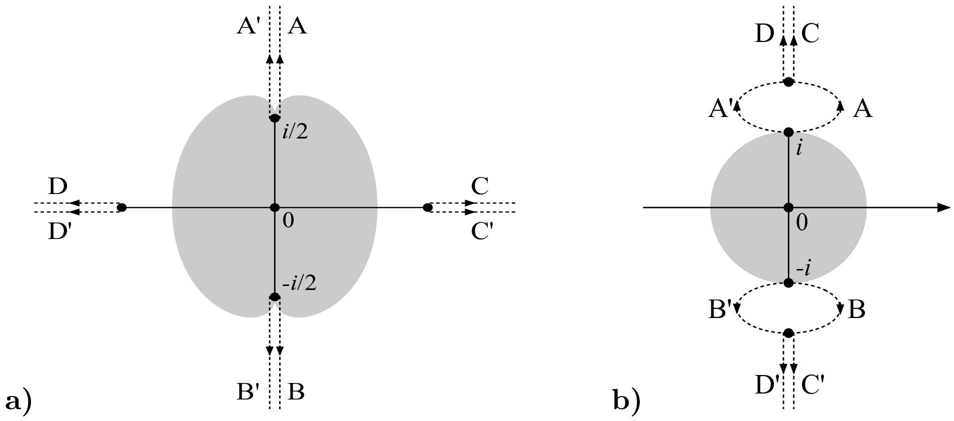

Figure 4: **a)** Representation of the surface state Ξ^α in the upper half plane, or rather, on the full plane using the doubling trick. The shaded region is the local coordinate, and 0 is the puncture. **b)** Picture of Ξ^α in the ξ coordinate, where the local coordinate is the unit disk. The contours A,A', etc. which are coincident in the UHP become separated in this representation because of branch cuts in f_α^{-1}. The surface must be cut along the dashed lines and glued A→A' etc. to get the appropriate surface.

The state Ξ depends implicitly on the choice of $f(\xi)$, and for the sliver frame Ξ is the $SL(2,\mathbb{R})$ vacuum. Beyond the above formal definition, Ξ and its powers Ξ^α are generally difficult to characterize analytically, for example as of correlators in the upper half plane. However, for a certain class of projector frames — the *special projectors* introduced in ref. [12]—the Ξ^αs can be constructed explicitly.

To make sense of these solutions, the first step should be to calculate the energy and see whether or not it reproduces the D-brane tension. The main challenge here is to give a simple definition of these solutions in CFT language, so that it is clear how to calculate the inner products necessary evaluate the energy in terms of CFT correlators. Our strategy will be to define a certain conformal frame, the *strip frame*, where the operator K_L^v becomes a Hamiltonian creating an infinitesimal strip of worldsheet. In this formulation the solutions look very similar to Schnabl's solution on the cylinder, as in figure 3. Moreover, for frames where $f(\xi)$ diverges only at the midpoint, it is manifest that the energy calculation proceeds exactly as for Schnabl's solution. Thus, these solutions are simply alternative descriptions of the closed string vacuum. Though we will not discuss it explicitly, in fact it turns out that the solutions are related by midpoint-preserving reparameterizations [10].

4.1 The strip frame

We begin our analysis by considering an explicit example: the butterfly projector, defined by the conformal frame,

$$f(\xi) = \frac{\xi}{\sqrt{1+\xi^2}} \tag{4.2}$$

This maps the canonical half disk $|\xi| < 1$ to a region in the upper half plane bounded by the branches of the hyperbola $x^2 - y^2 = \frac{1}{2}$. The vector field $v(\xi)$ and the operators defining the solution become,

$$\begin{aligned} v(\xi) &= \frac{1}{\xi}(1+\xi^2)^2 \\ K_L^v &= (L_2+2L_0+L_{-2})_L \quad B_L^v = (b_2+2b_0+b_{-2})_L \quad c^v(1) = -\frac{1}{4}c(1) \end{aligned} \tag{4.3}$$

As discussed in ref. [12], the butterfly is a *special projector*, meaning the energy-momentum zero modes $\mathcal{L}_0, \mathcal{L}_0^*$ in this frame satisfy the algebra,

$$[\mathcal{L}_0, \mathcal{L}_0^*] = s(\mathcal{L}_0 + \mathcal{L}_0^*) \tag{4.4}$$

where for the butterfly $s = 2$. This extra structure gives us a little more information about how to construct the analogue of wedge states for the butterfly, the Ξ^αs. Explicitly, they may be represented as surface states on the upper half plane, with a local coordinate [12]

$$\mu = f_\alpha(\xi) = \frac{\xi\sqrt{1+2\alpha(1+\xi^2)}}{1-\xi^2+2\alpha(1+\xi^2)} \tag{4.5}$$

Also useful is the inverse map,

$$\xi = f_\alpha^{-1}(\mu) = \sqrt{\frac{2\alpha+1}{2}}\left(\frac{1-2(2\alpha-1)\mu^2-\sqrt{4\mu^2-1}}{(2\alpha-1)^2\mu^2-2\alpha}\right)^{1/2} \tag{4.6}$$

A picture of a typical Ξ^α surface state is shown in figure 4, both on the upper half plane and in the ξ-presentation (where the local coordinate is the unit disk). In the ξ-presentation there are various cuts in the surface, one surrounding a pseudo-circular region just above the local coordinate and another further above on the imaginary axis. These cuts must be glued together as indicated before the Ξ^α surface is formed. Neither coordinate system, known from ref. [12], is well suited for calculating star products or vertices.

The description of Ξ^α simplifies somewhat in the butterfly frame, as shown in figure 5a. After transforming with eq. (4.2), the pseudo-circular region gets mapped to the exterior of a hyperbola $x^2 - y^2 = \frac{1}{2} + \alpha$, lying just outside of the butterfly local coordinate. The branches of the outer hyperbola must be sewn together to form a kind of warped cylinder. Though an improvement over figure 4, these hyperbolic contours are awkward for gluing. There is a related issue here: Unlike the case for the sliver, the operator K_L^v in the butterfly frame is not simply a "Hamiltonian" which creates an infinitesimal strip. Rather,

$$f\circ K_L^v = \int_{i\infty}^{-i\infty}\frac{dz}{2\pi i}\frac{1}{z}T(z) \tag{4.7}$$

This suggests that we should look for another conformal frame where K_L^v simply becomes $\int \frac{dz}{2\pi i}T(z)$. We will call this the "strip frame" $y = H(\xi)$. We want,

$$H^{-1}\circ\int\frac{dy}{2\pi i}T(y) = \int_L\frac{d\xi}{2\pi i}\frac{1}{H'(\xi)}T(\xi) = K_L^v \tag{4.8}$$

In other words,

$$\frac{d}{d\xi}H(\xi) = \frac{1}{v(\xi)} \tag{4.9}$$

This equation is easily solved in the situation at hand. The result is,

$$H(\xi) = \frac{1}{2}\frac{\xi^2}{1+\xi^2} = \frac{1}{2}f(\xi)^2 \tag{4.10}$$

That is, we square $f(\xi)$ and divide by 2. The resulting picture of Ξ^α is very simple, as shown in figure 5b. Concentrate for the moment on the image of the upper half plane. The hyperbola-shaped local coordinate gets mapped to the entire region $\Re(z) < \frac{1}{4}$, whose positive and negative boundaries are now straight vertical lines above and below the real axis. The oddly shaped regions between the the cut and local coordinate are mapped to rectangular strips of width $\alpha/2$ above and below the real axis. Star multiplication of Ξ^αs is clear: it simply changes the width of these strips.[10] Gluing the strips together and sewing the image of the lower half plane onto positive real axis, we get the surface pictured in 5c. This is almost a cylinder of circumference $\alpha + \frac{1}{2}$, except there are two semi-infinite planes inside the local coordinate folded in half and glued parallel to the axis, joining at the puncture.

One minor subtlety with this coordinate system is the presence of a curvature singularity at the puncture. In particular, inserting an off-shell vertex operator with $H(\xi)$ produces a divergent factor from the conformal transformation. Of course, this singularity is fictitious since in the end we must map back to the upper half plane to calculate the correlator anyway. But to be proper, we should define the surface state in the strip frame as a limit,

$$\langle \Xi^\alpha, \chi\rangle = \lim_{\xi\to 0}\langle H\circ\chi(\xi)\rangle_{X_{\alpha+\frac{1}{2}}} \tag{4.11}$$

where we denote,

$$X_{\alpha+\frac{1}{2}} = H\circ f_\alpha^{-1}\circ \mathrm{UHP},$$

the surface pictured in figure 5b,c.

It is a simple exercise to express the ψ_n's in the strip frame, as shown in figure 6:

$$\langle \psi_n', \chi\rangle = \lim_{\xi\to 0}\left\langle c\left(n+\frac{3}{4}\right)KBc\left(\frac{3}{4}\right)H\circ\chi(\xi)\right\rangle_{X_{n+\frac{3}{2}}} \tag{4.12}$$

with the K, B contour insertions as in eq. (2.40). Note that care must be taken to insert the operators on the proper branch. In eq. (4.12) and figure 6 we attached the entire strip to the positive boundary of the local coordinate, so no confusion arises. However, if we were to attach half the strip to the negative boundary, as in figure 5b, both cs would be located at $\frac{3}{4}$; of course, this does not give zero because the insertions are on different branches. In this coordinate system, it is easy to prove the equations of motion by following step-by-step the CFT derivation given in ref. [3]. Furthermore, it is clear that the inner products $\langle\psi_m', Q_B\psi_n'\rangle$ will be the same as for Schnabl's solution, since after we cut away the

[10] If we use the identification to bring the entire strip to the positive boundary of the local coordinate, we have chosen our conventions so that Ξ^α always corresponds to a region of width α in the strip frame. For wedge states, the strip frame is related to the sliver frame by a factor of $2/\pi$, so this is consistent with eq. (2.30).

JHEP05(2007)083

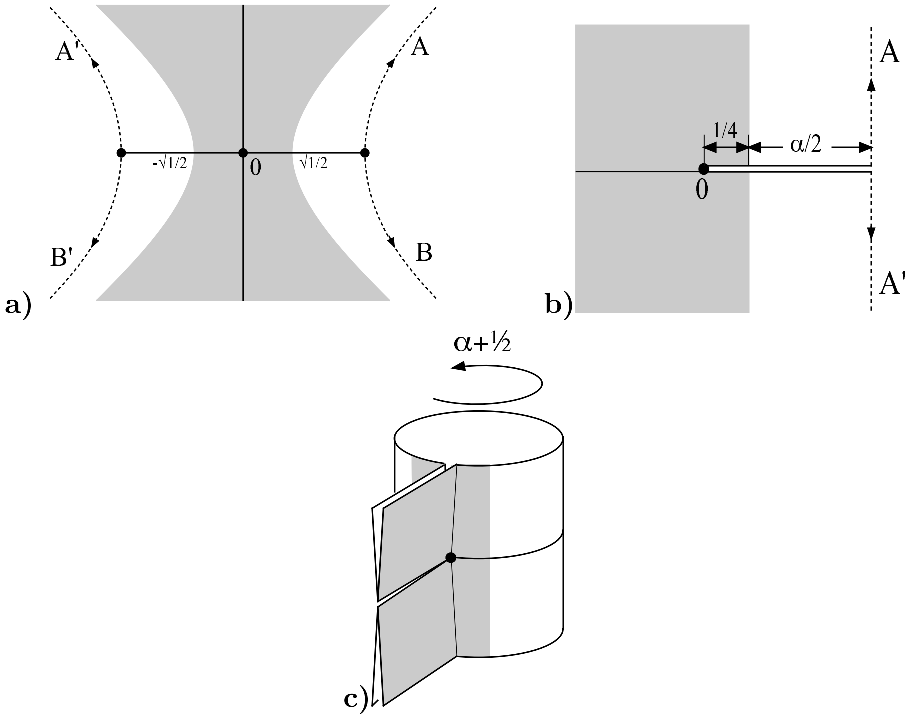

Figure 5: **a)** Representation of Ξ^α in the butterfly frame. The dashed hyperbolic branches A,A',B,B' are the images of the corresponding dashed pseudo-circles above and below the local coordinate in figure 4b. The surface must be cut and glued along these branches as indicated. **b)** The Ξ^αs in the strip frame, $H = \frac{1}{2}f^2$. For clarity we have only shown the image of the upper half plane. The semi-infinite grey shaded region $\Re(z) < \frac{1}{4}$ is the local coordinate, i.e. the image of the upper half unit semi-circle in the ξ-presentation. The dashed lines A,A' are the images of the hyperbolic branches in a) and should be glued together. The boundary of the open string in this picture is above and below the positive real axis, and if we use the doubling trick the image of lower half plane is on another Riemann sheet glued along the positive real axis. **c)** After gluing A,A' and including the lower half plane, we obtain a picture of a cylinder of circumference $\alpha + \frac{1}{2}$ together with two semi-infinite sheets folded in half and glued parallel to the axis of the cylinder and meeting at the puncture.

local coordinates and glue the strips together we are left with the same correlators on the cylinder. Therefore, the the butterfly solution reproduces the expected D-brane tension.

Let us discuss how these results generalize to other projector frames. It turns out that $H(\xi)$ satisfies an interesting reality condition which plays a particularly important role in this respect. Note that the reality condition for $v(\xi)$ eq. (2.20) implies:

$$\overline{\frac{d}{d\xi}H(\xi)} = \frac{1}{\overline{v(\xi)}} = \frac{1}{\bar{\xi}^2 v\left(\frac{1}{\bar{\xi}}\right)} = -\frac{d}{d\bar{\xi}}H\left(\frac{1}{\bar{\xi}}\right) \tag{4.13}$$

Integrating this yields,

$$\overline{H(\xi)} + H\left(\frac{1}{\bar{\xi}}\right) = 2A \tag{4.14}$$

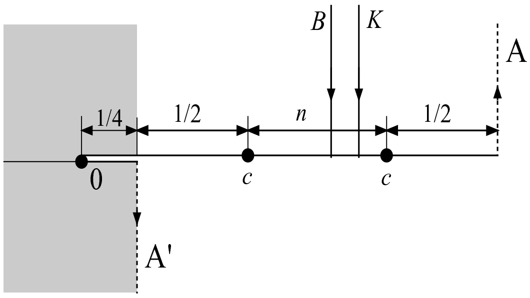

Figure 6: The ψ'_n building blocks of the butterfly generalization of Schnabl's solution. The entire strip has been glued to the positive boundary of the local coordinate, and only the image of the upper half plane is shown for clarity.

where A is a real integration constant. When $\xi = e^{i\theta}$ is on the unit circle, this implies:

$$\mathrm{Re} H(e^{i\theta}) = A \tag{4.15}$$

That is, the boundary of the local coordinate in the strip frame is always a straight, vertical line intersecting the real axis at A. Furthermore, because K_L^v creates pieces of worldsheet of constant width parallel to the imaginary axis, the states Ξ^α always correspond to strips of width α glued to the positive boundary of the local coordinate, regardless of the choice of projector. Therefore it is clear that the split string solution eq. (2.13) reproduces the correct D-brane tension regardless of the choice of vector field $v(\xi)$.[11] However for completely general $v(\xi)$ the local coordinate can be an extremely complicated Riemann surface, subject only to the constraint that its boundary is a vertical line in the complex plane. Thus it is difficult to perform explicit calculations with the solution, for example in level truncation.

For special projectors, however, the story drastically simplifies. Special projectors satisfy some additional constraints which yield a simple algebraic relationship between $H(\xi)$ and the coordinate frame defining the projector. As explained in ref. [12], all special conformal frames $f(\xi)$ satisfy the constraint,

$$\left[-f\left(-\frac{1}{\xi}\right)\right]^s + C_1 f(\xi)^s = C_2 \tag{4.16}$$

where C_1 and C_2 are constants over the unit circle, except possibly at points where $f(\xi)$ becomes singular. We can use this to simplify $v(\xi)$,

$$\begin{aligned} v(\xi) &= \frac{f(\xi)}{f'(\xi)} - \frac{f(-1/\xi)}{\frac{d}{d\xi} f(-1/\xi)} \\ &= \frac{f}{f'} - \frac{(C_2 - C_1 f^s)^{1/s}}{\frac{d}{d\xi}(C_2 - C_1 f^s)^{1/s}} \\ &= \frac{C_2}{C_1 f^{s-1} f'} = \frac{1}{H'} \end{aligned} \tag{4.17}$$

[11] modulo some singularities which we consider in the next subsection

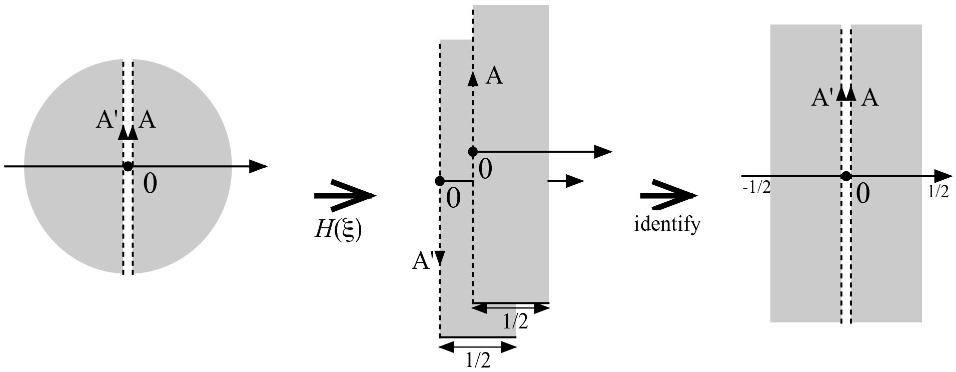

Figure 7: Mapping the local coordinate using the sliver's $H(\xi)$. The two halves of the unit disk get mapped to two copies of the strip $\Re(z) \in [0, \frac{1}{2}]$. The contours A,A' which are coincident on the disk must be glued so that the arrows point in the same direction. The result is the familiar picture of the sliver frame as a strip $\Re(z) \in [-\frac{1}{2}, \frac{1}{2}]$.

This equation integrates easily to give

$$H(\xi) = \frac{C_1}{C_2 s} f(\xi)^s, \tag{4.18}$$

a generalization of eq. (4.10).

Let us see how eq. (4.18) determines the structure of the local coordinate in the strip frame. Using constraints on C_1, C_2 derived in ref. [12] we can show,

$$\mathrm{Re}H(e^{i\sigma}) = \frac{1}{2s} \tag{4.19}$$

So the boundary of the local coordinate lies on a straight vertical line intersecting the real axis at $\frac{1}{2s}$. In fact, because the local coordinate reaches infinity at the midpoint, as we move around the unit circle from $\sigma = 0$ to 2π we will pass over this vertical line at least[12] twice. This means that the local coordinate patch lies on two or more Riemann sheets with boundary on this line — we have seen this phenomenon already for the butterfly. Next, note that the "constants" C_1, C_2 can be discontinuous on the unit circle where $f(\xi)$ has singularities. We may extend the definition of C_1, C_2 for $|\xi| < 1$ by defining them to be piecewise constant on pie shaped regions inside the unit circle. These discontinuities then imply that the local coordinate patch in the strip frame generally has cuts extending from the origin (the puncture) to infinity, with implicit identifications. The simplest example of this is the sliver $f(\xi) = \tan^{-1}\xi$, where,

$$C_1 = 1 \quad C_2 = \frac{\pi}{2}\epsilon(\xi) \tag{4.20}$$

and $\epsilon(\xi)$ is the step function, $= 1$ for $\mathrm{Re}(\xi) > 0$ and -1 for $\mathrm{Re}(\xi) < 0$. Plugging in to eq. (4.18) then gives,

$$H(\xi) = \frac{2}{\pi}\epsilon(\xi)\tan^{-1}\xi \tag{4.21}$$

[12] If $f(\xi)$ has singularities elsewhere than at the midpoint, there may be many copies of this line — see next section.

JHEP05(2007)083

This maps the unit disk to two copies of the strip $\mathrm{Re}(z) \in [0, \frac{1}{2}]$, as shown in figure 7. Note that the local coordinate lies on two Riemann sheets, as anticipated above. On the negative boundaries of these strips are cuts extending from the origin to infinity; these cuts are images of of the contours on either side of the imaginary axis on the disk. Sewing these cuts together, we recover the usual picture of the sliver local coordinate (up to a scaling) as a single strip, $\mathrm{Re}(z) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

4.2 Multi-winged butterflies

Let us give another class of examples which seem fundamentally different from the sliver and butterfly. We will call these "multi-winged" butterflies, and they are defined by the conformal frames [13, 12]:

$$f_{(m)}(\xi) = \frac{\xi}{(1+(-1)^{m+1}\xi^{2m})^{1/2m}} \tag{4.22}$$

for integer $m \geq 1$ and have $s = 2m$; $m = 1$ is the butterfly. Calculating $v(\xi)$ and the strip frame,

$$\begin{aligned} v_{(m)}(\xi) &= \xi\left[2+(-1)^{m+1}(\xi^{2m}+\xi^{-2m})\right] \\ H_{(m)}(\xi) &= \frac{1}{2m}\frac{\xi^{2m}}{\xi^{2m}+(-1)^{m+1}} = \frac{(-1)^{m+1}}{2m}[f^m(\xi)]^{2m} \end{aligned} \tag{4.23}$$

consistent with the general structure eq. (4.18). We can also work out the image of the unit circle in the strip frame,

$$\begin{aligned} H_{(m)}(e^{i\theta}) &= \frac{1+i\tan m\theta}{4m} \quad m \text{ odd} \\ &= \frac{1-i\cot m\theta}{4m} \quad m \text{ even} \end{aligned} \tag{4.24}$$

The image curves have constant real part $1/4m$ and so are straight lines, as expected. The interesting thing about these frames is that when θ increases over the left half of the string, the coordinate passes through infinity and over the same vertical line many times. This is a consequence of the fact that the frames are singular at multiple points on the unit circle,

$$\theta_k = \frac{k}{m}\frac{\pi}{2} \tag{4.25}$$

where k is odd when m is odd and k is even when m is even. The appearance of many copies of the same vertical line indicates that we should think of the local coordinate in terms of multiple Riemann sheets. The states Ξ^α are obtained by gluing strips of length $\alpha/2$ to the branches of the boundary of the local coordinate, as shown in figure 8. The situation is analogous to the butterfly, only now we have "many wings."

Let us now try to make solutions in these frames. Note that when m is even, an odd thing happens: the boundary of the open string is mapped to $-i\infty$. When constructing ψ_ns we need to place c ghosts on the boundary, so this requires ghost insertions in the strip frame at $-i\infty$. We can see a related problem in the split string formalism. When m

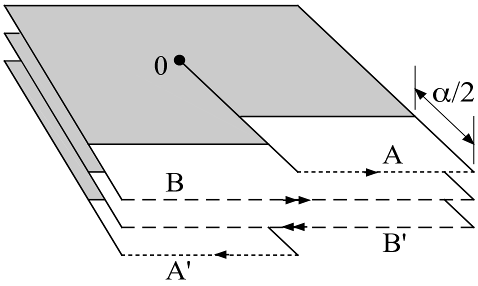

Figure 8: Riemann sheets of the state Ξ^α in the strip frame for the $m = 3$ butterfly. Shaded in grey is the local coordinate, and in white are the attached strips. The topmost and bottom most sheets A,A' must be identified so that the arrows point in the same direction. Likewise, the middle two sheets B,B' must be identified so that the double arrows point in the same direction. This picture only shows the image of the upper half plane; there is a similar picture for the lower half plane which must be glued to the line segments representing the boundary of the open string. This gives a total of 6 complete branches.

is even $v(1) = 0$, so the corresponding c field is divergent. Perhaps these solutions can be regulated by temporarily relaxing reality and taking the ghosts off the real axis.

When m is odd, however, the solutions seem better, at least in the Fock space. Take for example the ψ_n's when $m = 3$. Moving everything to the positive boundary of the local coordinate, we obtain three strips of width $n + 1$. One strip is bisected by the open string boundary; we place c insertions and K, B contours inside exactly as in figure 6. The other two strips contain the images of the unit circle for $\theta \in [\frac{\pi}{6}, \frac{\pi}{2}]$ and $\theta \in [-\frac{\pi}{2}, -\frac{\pi}{6}]$, respectively; these strips contain the remaining pieces of the K, B contours. Calculation of the energy, however, brings a puzzling phenomenon: upon removing the local coordinate, the inner product $\langle \psi_n', Q_B \psi_n' \rangle$ involves a correlator on three disconnected cylinders! We were not able to follow the analysis far enough to come to a definite conclusion about these solutions, but there could be an interesting phenomenon here.

5. Conclusion

In this paper we have explored some generalizations of Schnabl's solution suggested by the split string formalism. All of the solutions we have analyzed turn out to be related by midpoint preserving reparameterizations [10]; indeed, with the help of the strip frame eq. (4.9) it is manifest that these solutions are physically equivalent. Less obvious is what happens when we let F be a completely general element of the Abelian algebra of wedge states; we will consider this question in the companion paper (II).

Originally our hope was that the split string solution eq. (2.13) would be general enough to accommodate multiple brane vacua, though this possibility seems not to have been realized. It is possible that solutions based on multi-branched projector frames could bring surprises, or perhaps a completely new understanding of the ψ_N piece is necessary. Another possible approach would be to explore the vacuum string field theory limit, where multiple brane solutions are better understood.

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